Sensor position estimation

I. Introduction

Based on field reconstruction [1], the problem formulated to predict the position of the sensors has been described.

II. Problem Formulation

Riemann approximation of the Fourier coefficients with the noise corrupted sample values,

$$\hat{A}_{gen}[k] = \frac{1}{M}\sum_{i=1}^{M} g^o(s_i)e^{-j2\pi ki/M} \quad (1)$$

Thus reconstructing the field using these,

$$\begin{aligned}
\hat{g}(s_l) &= \sum_{k=-b}^{k=b} \hat{A}_{gen}[k]e^{jk2\pi s_l} \\
&= \sum_{k=-b}^{k=b} e^{jk2\pi s_l}\left(\frac{1}{M}\sum_{i=1}^{M} g^o(s_i)e^{-j2\pi k\frac{i}{M}}\right) \\
&= \frac{1}{M}\sum_{k=-b}^{k=b}\left(\sum_{i=1}^{M} g^o(s_i)e^{j2\pi k(s_l-\frac{i}{M})}\right) \\
&= \frac{1}{M}\sum_{i=1}^{M}\left(\sum_{k=-b}^{k=b} g^o(s_i)e^{j2\pi k(s_l-\frac{i}{M})}\right) \\
&= \frac{1}{M}\sum_{i=1}^{M}\left(g^o(s_i)\frac{e^{-j2\pi b(s_l-\frac{i}{M})}(e^{j2\pi(2b+1)(s_l-\frac{i}{M})}-1)}{e^{j2\pi(s_l-\frac{i}{M})}-1}\right) \quad (2)
\end{aligned}$$

Bringing it to a compact form,

$$\hat{g} = Fg^o \quad (3)$$

where,

$$F = \begin{bmatrix} \frac{e^{-j2\pi b(s_1-\frac{1}{M})}(e^{j2\pi(2b+1)(s_1-\frac{1}{M})}-1)}{e^{j2\pi(s_1-\frac{1}{M})}-1} & \dots & \frac{e^{-j2\pi b(s_1-\frac{M}{M})}(e^{j2\pi(2b+1)(s_1-\frac{M}{M})}-1)}{e^{j2\pi(s_1-\frac{M}{M})}-1} \\ . & \dots & . \\ . & \dots & . \\ \frac{e^{-j2\pi b(s_M-\frac{1}{M})}(e^{j2\pi(2b+1)(s_M-\frac{1}{M})}-1)}{e^{j2\pi(s_M-\frac{1}{M})}-1} & \dots & \frac{e^{-j2\pi b(s_M-\frac{M}{M})}(e^{j2\pi(2b+1)(s_M-\frac{M}{M})}-1)}{e^{j2\pi(s_M-\frac{M}{M})}-1} \end{bmatrix},$$

and

$$\bar{g^o} = \begin{bmatrix} g^o(s_1) \\ g^o(s_2) \\ . \\ . \\ g^o(s_M) \end{bmatrix}$$

Thus, the optimisation problem takes the form,

$$\begin{aligned}
\bar{s} : \underset{\bar{s}}{\text{minimize}} & \left\|\frac{1}{M}F\bar{g^o} - \bar{g^o}\right\|_2^2 \\
\text{subject to} & \\
& 0 \le s_1 \le s_2 \le \dots \le s_{M-1} \le s_M
\end{aligned} \quad (4)$$

III. Complete solution without assumptions

The original problem in 4 had each variable bounded by the other variables, restated here for convenience,

$$0 \le s_1 \le s_2 \dots \le s_{M-1} \le s_M$$

The above inequalities can be recast as,

$$\begin{aligned}
a_1 &= s_1, \quad a_1 > 0 \\
a_2 &= s_2 - s_1, \quad a_2 > 0 \\
a_3 &= s_3 - s_2, \quad a_3 > 0 \\
&. \\
&. \\
&. \\
a_M &= s_M - s_{M-1}, \quad a_M > 0
\end{aligned} \quad (5)$$

Therefore, the problem can be stated as,

$$\begin{aligned}
\bar{s} : \underset{\bar{s}}{\text{minimize}} & \left\|\frac{1}{M}F(s)\bar{g^o} - \bar{g^o}\right\|_2^2 \\
\text{subject to} & \\
& A - T\bar{s} = 0 \\
& T\bar{s} \ge 0 \\
& \bar{s} \ge 0
\end{aligned} \quad (6)$$

where, $A = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ a_M \end{bmatrix}$, and

from 5, $T = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ . & \dots & . & & & \\ . & \dots & . & & & \\ . & \dots & . & & & \\ 0 & 0 & 0 & \dots & -1 & 1 \end{bmatrix}$

IV. Reformulation

Rewriting the optimization problem in (**??**) in terms of the independent variables. As is defined in (5), we can equivalently write,

$$\begin{aligned} &s_1 = a_1, \quad s_1 > 0 \\ &s_2 = a_1 + a_2, \quad s_2 > 0 \\ &s_3 = a_1 + a_2 + a_3, \quad s_3 > 0 \\ &. \\ &. \\ &s_i = a_1 + a_2 + a_3 + ... + a_i, \quad a_i > 0 \\ &\forall i \in [1, M] \end{aligned} \tag{7}$$

The elements of F-matrix in (II) can be written as

$$F_{i,k} = \frac{e^{-j2\pi b(s_i - \frac{k}{M})}(e^{j2\pi(2b+1)(s_i - \frac{k}{M})} - 1)}{e^{j2\pi(s_i - \frac{k}{M})} - 1} \tag{8}$$

Rewriting in terms on 7, we have

$$F_{i,k} = \frac{e^{-j2\pi b(a_1+a_2+...+a_i - \frac{k}{M})}(e^{j2\pi(2b+1)(a_1+a_2+...+a_i - \frac{k}{M})} - 1)}{e^{j2\pi(a_1+a_2+...+a_i - \frac{k}{M})} - 1} \tag{9}$$

Therefore, the optimization problem can now be written as,

$$\begin{aligned} \bar{a} : \underset{\bar{a}}{\text{minimize}} & \left\| \frac{1}{M} F(a)\bar{g^o} - \bar{g^o} \right\|_2^2 \\ \text{subject to} & \\ & \bar{s} - L\bar{a} = 0 \\ & \bar{s} \geq 0 \\ & \bar{a} \geq 0 \end{aligned} \tag{10}$$

where L is a lower triangular matrix of size MxM.

V. Corrected version

The previous formation of matrix had assumed the constant difference of $\frac{1}{M}$ between the sampling point which will not be so for randomly generated sampling points.. Therefore, for estimation of Fourier coefficients now we have

$$\hat{A}_{gen}[k] = s_1 g^o(s_1)e^{-j2\pi k s_1} + \sum_{i=2}^{M}(s_i - s_{i-1})g^o(s_i)e^{-j2\pi k s_i} \tag{11}$$

Accordingly, the closed form equation of the reconstructed samples would be

$$\begin{aligned} \hat{g}(s_l) &= \sum_{k=-b}^{k=b} \hat{A}_{gen}[k]e^{jk2\pi s_l} \\ &= \sum_{k=-b}^{k=b}\Bigg((s_1 g(s_1)e^{j2\pi k(s_l - s_1)}) + (\sum_{i=2}^{M}(s_i - s_{i-1})g(s_i) \\ &\quad e^{j2\pi k(s_l - s_i)}))\Bigg) \\ &= s_1 g(s_1)\frac{\sin(\pi(2b+1)(s_l - s_1))}{\sin(\pi(s_l - s_1))} + \sum_{i=2}^{M}\Bigg((s_i - s_{i-1}) \\ &\quad g(s_i)\frac{\sin(\pi(2b+1)(s_l - s_i))}{\sin(\pi(s_l - s_i))}\Bigg) \end{aligned} \tag{12}$$

Rewriting in a compact form

$$\hat{g}(s) = \mathcal{F}\mathcal{S}g(s) \tag{13}$$

where,

$$\mathcal{F} = \begin{bmatrix} F(1,1) & F(1,2) & F(1,3) & ... & F(1,M) \\ F(2,1) & F(2,2) & F(2,3) & ... & F(2,M) \\ . & . & . & ... & \\ . & . & . & ... & \\ . & . & . & ... & \\ F(M,1) & F(M,2) & F(M,3) & ... & F(M,M) \end{bmatrix},$$

In accordance with 13

$$F(i,j) = \frac{\sin(\pi(2b+1)(s_i - s_j))}{\sin(\pi(s_i - s_j))} \tag{14}$$

For elements at $i = j$, the limit is calculated to be (2b+1). And given the elements F, it can be seen that V is symmetric. And,

$$\mathcal{S} = \begin{bmatrix} s_1 & 0 & 0 & ... & 0 \\ 0 & s_2 - s_1 & 0 & ... & 0 \\ 0 & 0 & s_3 - s_2 & ... & 0 \\ & & . & & \\ & & . & & \\ 0 & 0 & 0 & ... & s_M - s_{M-1} \end{bmatrix}$$

To remove the dependency of variables, adopting the method shown in (7), redefining (14) as

$$F(i,j) = \frac{\sin(\pi(2b+1)(a_1 + a_2 + ... + a_i - a_1 - a_2 + ... - a_j))}{\sin(\pi(a_1 + a_2 + ... + a_i - a_1 - a_2 + ... - a_j))} \tag{15}$$

Considering that $\mathcal{F}$ is symmetric and thus can be expressed such as $i > j$ always, we can rewrite the above expression for F as

$$F(i,j) = \frac{\sin(\pi(2b+1)(a_{j+1} + a_{j+2} + ... + a_i))}{\sin(\pi(a_{j+1} + a_{j+2} + ... + a_i))} \tag{16}$$

$$\text{And, } \mathcal{S} = \begin{bmatrix} a_1 & 0 & 0 & ... & 0 \\ 0 & a_2 & 0 & ... & 0 \\ 0 & 0 & a_3 & ... & 0 \\ & & . & & \\ & & . & & \\ 0 & 0 & 0 & ... & a_M \end{bmatrix}$$

Finally, the optimization problem can be solved as

$$\begin{aligned} \bar{a} : \underset{\bar{a}}{\text{minimize}} & \|\mathcal{F}\mathcal{S}\bar{g^o} - \bar{g^o}\|_2^2 \\ \text{subject to} & \\ & \bar{a} \geq 0 \end{aligned} \tag{17}$$

where L is a lower triangular matrix of size MxM.

VI. Results for lower samples

Each simulation was run 20 times. Expression for the error functions:-

$$\mathcal{O} = \frac{average(\|g_{rec} - g\|_2^2)}{n} \tag{18}$$

$$\mathcal{E} = \frac{\mathbb{E}[|a_{recon}(k) - a(k)|^2]}{n} \tag{19}$$

$\mathcal{E}$ averaged over number of simulations (20 in this case).

$$\mathcal{P} = \frac{\|s_{true} - s_{rec}\|_2^2}{n} \tag{20}$$

$\mathcal{P}$ averaged over 20 simulations. The following results are for noiseless case.

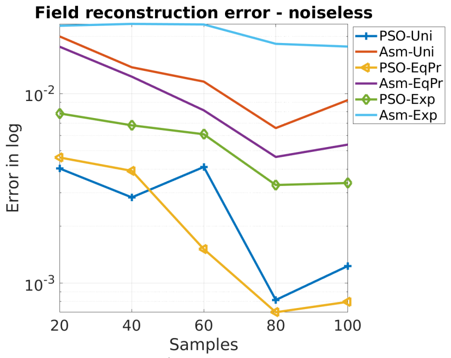

Fig. 1

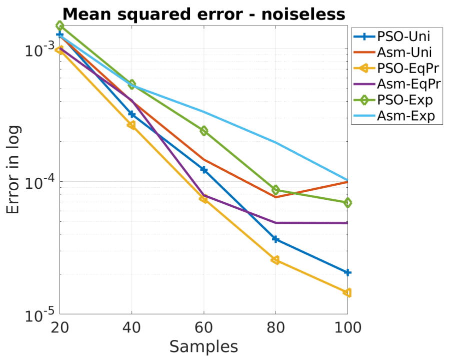

Fig. 2

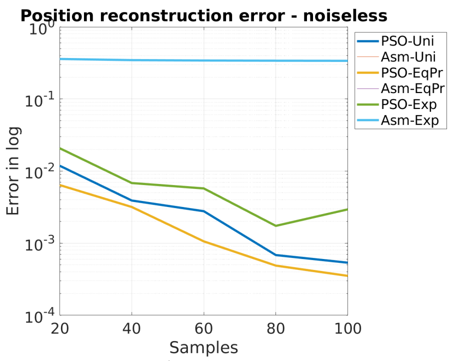

Fig. 3

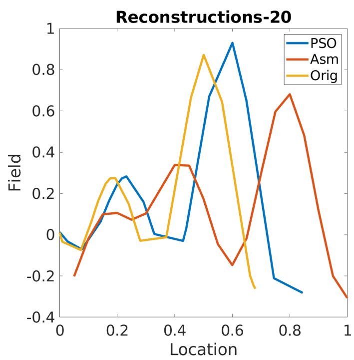

Fig. 4

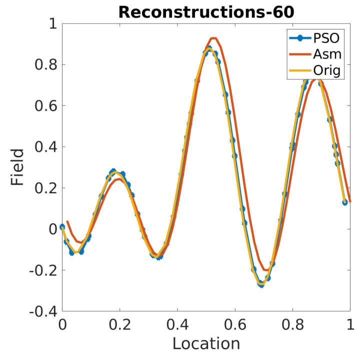

Fig. 5

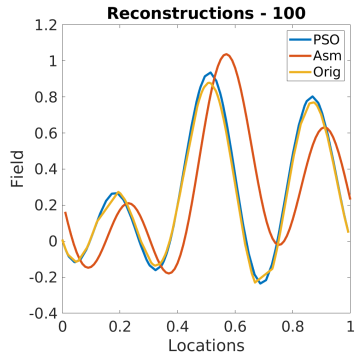

Fig. 6

The following results are for the noisy case.

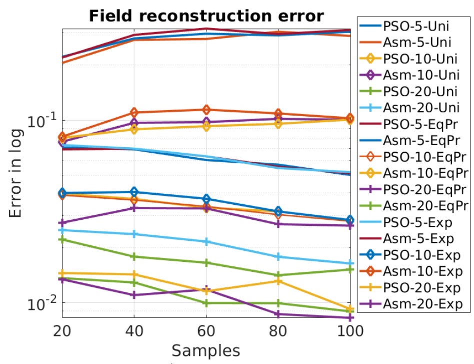

Fig. 7

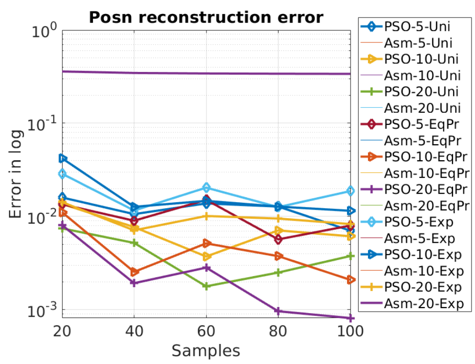

Fig. 8

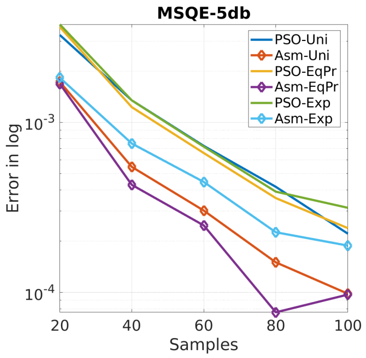

Fig. 9

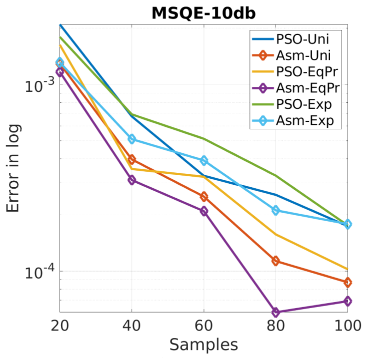

Fig. 10

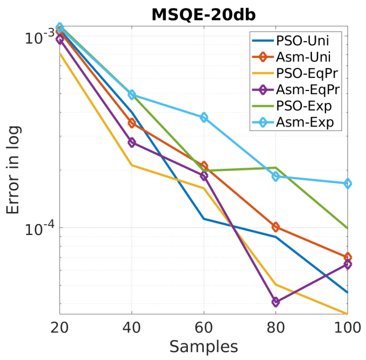

Fig. 11

VII. Results using NN

A neural network was implemented as the problem was treated as a nonlinear regression problem. The neural network was trained with the magnitude of the field as input and the sampling locations as output from the data generated by three different distributions and varying SNR (noiseless, 5, 10 and 20dB).
The neural network was made up of 3 hidden layers and the input and output layers.Other parameters were varied to land up at the best result and they have been mentioned in the caption.The reconstruction results are presented below.

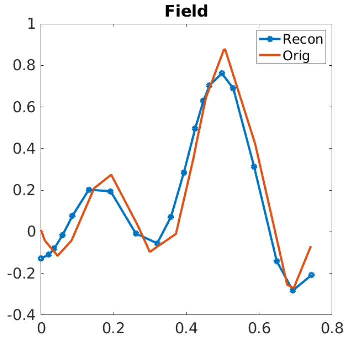

Fig. 12: Noiseless, Reconeror:1.9%, batchsize:10, epoch:50, N:100,60,60

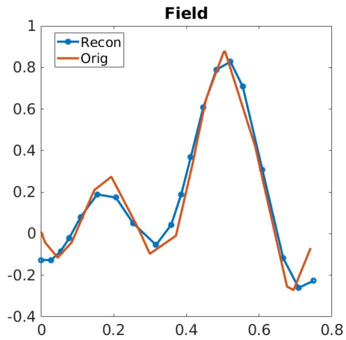

Fig. 13: Noiseless, Reconeror:2.4%, batchsize:10, epoch:50, N:60,100,100

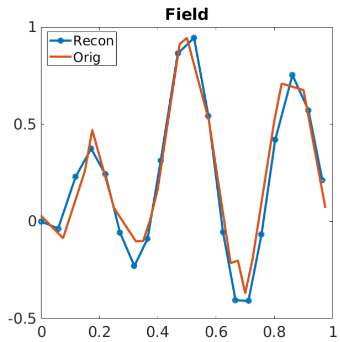

Fig. 14: Noisy, Reconeror:8.4%, batchsize:10, epoch:50, N:60,100,100

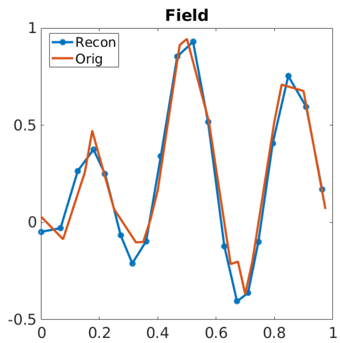

Fig. 15: Noisy, Reconeror:8.2%, batchsize:100, epoch:50, N:100,60,60

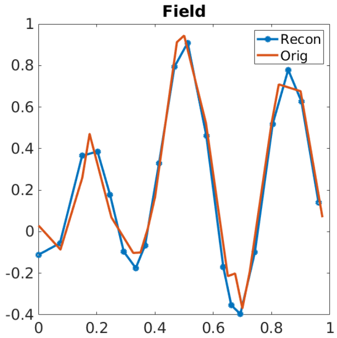

Fig. 16: Noisy, Reconeror:8.12%, batchsize:1000, epoch:500, N:120,200,200

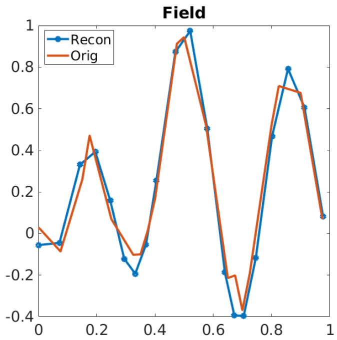

Fig. 17: Noisy, Reconeror:8.05%, batchsize:1000, epoch:500, N:120,200,200,testsize=0.1

References

[1] A. Kumar, "On bandlimited field estimation from samples recorded by a location-unaware mobile sensor," *IEEE Transactions on Information Theory*, vol. 63, no. 4, pp. 2188 – 2200, 2017.